

Coming back, either $H_z = 0$ or $E_z = 0$

if $H_z = 0$ - TM Wave $\rightarrow$ Mag field is purely transverse

$E_z = 0$ - TE wave $\rightarrow$ Elec field is purely transverse

Let revisit eqn 3 & 6

$$\frac{1}{r} \frac{\partial (r E_\phi)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} = -\mu j \omega H_z$$

$$\frac{1}{r} \frac{\partial (r H_\phi)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \phi} = \epsilon j \omega E_z$$

And the phasor form of H_ϕ H_r E_z E_ϕ eqns for wave travelling in z -direction.

$$E_r = \frac{-j}{k_c^2} \left[\frac{\omega \mu}{r} \frac{\partial H_z}{\partial \phi} + \beta \frac{\partial E_z}{\partial r} \right]$$

$$E_\phi = \frac{-j}{k_c^2} \left[\frac{\beta}{r} \frac{\partial E_z}{\partial \phi} - \omega \mu \frac{\partial H_z}{\partial r} \right]$$

$$H_\phi = \frac{-j}{k_c^2} \left[\frac{\beta}{r} \frac{\partial H_z}{\partial \phi} + \epsilon \omega \frac{\partial E_z}{\partial r} \right]$$

$$H_r = \frac{-j}{k_c^2} \left[\beta \frac{\partial H_z}{\partial r} - \frac{\epsilon \omega}{r} \frac{\partial E_z}{\partial \phi} \right]$$

Let's derive eqns for TM wave

Replace $H_z = 0$

$$\frac{1}{r} \frac{\partial (r E_\phi)}{\partial r} = \frac{1}{r} \frac{\partial E_r}{\partial \phi} \quad - (3)a$$

$$\frac{1}{r} \frac{\partial (r H_\phi)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \phi} = \epsilon j \omega E_z \quad - (4)a$$

$$E_r = \frac{-j}{k_c^2} \beta \frac{\partial E_z}{\partial r}$$

$$E_\phi = \frac{-j}{k_c^2} \frac{\beta}{r} \frac{\partial E_z}{\partial \phi}$$

$$H_\phi = \frac{-j}{k_c^2} \epsilon \omega \frac{\partial E_z}{\partial r}$$

$$H_r = \frac{j}{k_c^2} \frac{\epsilon \omega}{r} \frac{\partial E_z}{\partial \phi}$$

And $\frac{\partial}{\partial r} H_\phi$ & $\frac{\partial H_r}{\partial \phi}$ & substitute in 6a

$$H_\phi = \frac{-j}{k_c^2} \epsilon \omega \frac{\partial E_z}{\partial r} \quad r H_\phi = \frac{-j r \epsilon \omega}{k_c^2} \frac{\partial E_z}{\partial r}$$

Partial differentiation w.r.t r

$$\frac{\partial (r H_\phi)}{\partial r} = \frac{-j \epsilon \omega}{k_c^2} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right)$$

$$H_r = \frac{j}{k_c^2} \frac{\epsilon \omega}{r} \frac{\partial E_z}{\partial \phi} \quad \frac{\partial H_r}{\partial \phi} = \frac{j}{k_c^2} \frac{\epsilon \omega}{r} \frac{\partial^2 E_z}{\partial \phi^2}$$

$$6a \rightarrow \frac{1}{r} \times \frac{-j}{k_c^2} \epsilon \omega \left(\frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) \right) - \frac{1}{r} \times \frac{j}{k_c^2} \frac{\epsilon \omega}{r} \frac{\partial^2 E_z}{\partial \phi^2}$$

$$\frac{-j \epsilon \omega}{r k_c^2} E_z = \frac{-j \epsilon \omega}{r k_c^2} \left[\frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) - \frac{1}{r} \frac{\partial^2 E_z}{\partial \phi^2} \right]$$

$$E_z = -\frac{1}{k_c^2} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} \right]$$

Scalar helmholtz
eqn is independent
of time!

↓ observe it has pattern of laplacian in cylindrical coordinates

This appears like a scalar helmholtz
eqn form — in cylindrical coordinates.

$$\nabla^2 \psi + k^2 \psi = 0$$

$$\psi(x, y, z)$$

↓

$$\psi(r, \phi, z)$$

laplacian in cylindrical coordinates look like this

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Let's do detail observation before proceeding further

$$-k_c^2 E_z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} \quad \text{--- (A)}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = -k^2 \psi \quad \text{--- (B)}$$

Remember we assumed $\frac{\partial}{\partial z} \rightarrow -j\beta$ for travelling wave

$$\text{Then } \frac{\partial E_z}{\partial z} \rightarrow -j\beta E_z$$

$$\frac{\partial^2 E_z}{\partial z^2} \rightarrow -j\beta \frac{\partial E_z}{\partial z}$$

$$\rightarrow -j\beta (-j\beta)$$

$$\rightarrow \underline{-\beta^2}$$

$$k_c^2 = \omega_c^2 - \beta^2$$

From Eqn A $\rightarrow -\beta^2$ is absorbed into $k_c^2 E_z$ term

$$\text{(B) rewritten } \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} = -k^2 \psi - \frac{\partial^2 \psi}{\partial z^2}$$

$$\text{A} \rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} = -\omega_c^2 E_z + \beta^2 E_z$$

$\downarrow$
 $-\frac{\partial^2 E_z}{\partial z^2} \text{ term}$

Now let's go ahead & solve this PDE

We'll be using Variable Separable method to convert PDE to ODE

$$\text{Let } E_z(r, \phi, z) = R(r) \Phi(\phi) \cancel{Z(z)}$$

$$\frac{\partial}{\partial r} E_z(r, \phi) = \left[\frac{\partial}{\partial r} R(r) \right] \times \bar{\Phi}(\phi)$$

$$\frac{\partial}{\partial \phi} E_z(r, \phi) = R(r) \times \frac{\partial}{\partial \phi} [\bar{\Phi}(\phi)]$$

$$\frac{\partial^2}{\partial \phi^2} E_z(r, \phi) = R(r) \times \frac{\partial^2}{\partial \phi^2} [\bar{\Phi}(\phi)]$$

Substitute these in Eqn A

$$-k_c^2 [R(r) \bar{\Phi}(\phi)] = \frac{1}{r} \frac{\partial}{\partial r} \left[r \times \bar{\Phi}(\phi) \frac{d}{dr} R(r) \right] +$$

$$\frac{1}{r^2} R(r) \times \frac{d^2}{d\phi^2} (\bar{\Phi}(\phi))$$

$[R(r) \rightarrow R \quad \bar{\Phi}(\phi) \rightarrow \bar{\Phi}]$

$$-k_c^2 R \bar{\Phi} = \frac{1}{r} \times \bar{\Phi} \frac{d}{dr} \left[r \frac{d}{dr} R \right] + \frac{R}{r^2} \frac{d^2}{d\phi^2} \bar{\Phi}$$

$$-k_c^2 = \frac{1}{rR} \frac{d}{dr} \left[r \frac{d}{dr} R \right] + \frac{1}{r^2 \bar{\Phi}} \frac{d^2}{d\phi^2} \bar{\Phi}$$

$$-k_c^2 r^2 = \frac{r}{R} \frac{d}{dr} \left[r \frac{d}{dr} R \right] + \frac{1}{\bar{\Phi}} \frac{d^2 \bar{\Phi}}{d\phi^2}$$

$$\frac{r}{R} \frac{d}{dr} \left[r \frac{d}{dr} R \right] + k_c^2 r^2 = - \frac{1}{\bar{\Phi}} \frac{d^2 \bar{\Phi}}{d\phi^2}$$

Right side is a function of ϕ containing angles (azimuth value)

Left side has radial values r

The only way $\bar{\Phi}$ & R functions are equal is that they are equal to some constant.

Let the constant be $n^2 \rightarrow$ why square term? Making double differentiation simple? avoiding Γ term in solution

Let's observe

$$-\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = \text{constant}$$

The solution of this / roots of this ODE

W.G allows sine/cos waves inside $\Rightarrow$ the roots of above ODE should have $\pm j$ const terms. ~~Also~~

Let's solve this ODE

$$-\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = n^2$$

$$\frac{d^2 \Phi}{d\phi^2} = -n^2 \Phi$$

$$\frac{d^2 \Phi}{d\phi^2} + n^2 \Phi = 0 \quad \text{--- Linear homogenous D.E with const coeff}$$

~~$$\frac{d^2 \Phi}{d\phi^2} = 0 \quad \frac{d}{d\phi} = 0$$~~

~~$$D^2 + n^2$$~~

Solving this using standard method in maths

Observe the differential eqn. The double differentiation of a function is equal to const times the function.

This happens for a function like e^x , $\cos x$, $\sin x$

e^x is unbounded has ' ∞ ' so wave cannot vary in this way.

So, the function has to be a sinusoidal.

Should it be $A \sin x$ or $A \cos x$ or $A \sin x + B \cos x$?

If solution is $A \sin x$ or $A \cos x$

the double differentiation gives $A \sin x \rightarrow -A \sin x$

Peaks at $\frac{n\pi}{2}$ allowed for $A \sin x = \phi \rightarrow$

Peaks at $n\pi$ allowed for $A \cos x$

To have peaks at arbitrary angles also, the solution should have both $\cos$ & $\sin$ terms

So the Φ can be a form of $A \cos(n\phi) + B \sin(n\phi)$

$n \rightarrow$ comes out when differentiated $n \rightarrow n^2$ when double differentiated

$$\text{Let } \Phi(\phi) = A \cos(n\phi) + B \sin(n\phi)$$

$$\frac{d}{d\phi} \Phi = -An \sin(n\phi) + Bn \cos(n\phi)$$

$$\frac{d^2}{d\phi^2} \Phi = -An^2 \cos(n\phi) - Bn^2 \sin(n\phi)$$

$$\therefore -An^2 \cos(n\phi) - Bn^2 \sin(n\phi) = -n^2 [A \cos(n\phi) + B \sin(n\phi)]$$

$$\Rightarrow \cancel{-An^2 \cos n\phi} = \cancel{-Bn^2 \sin n\phi} \quad \underline{\text{LHS} = \text{RHS}}$$

~~$\sin$ & $\cos$ are equal~~

So we can finalize $\Phi(\phi) = A \cos(n\phi) + B \sin(n\phi)$

Now let's go to the Radial term — Bessel's equation — Bessel function

$$\frac{r}{R} \frac{d}{dr} \left[r \frac{d}{dr} R \right] + k_c^2 r^2 = n^2$$

$$r \frac{d}{dr} \left[r \frac{d}{dr} R \right] + k_c^2 r^2 R - n^2 R = 0$$

$$r \frac{d}{dr} \left[\underbrace{r}_{u} \frac{d}{dr} \underbrace{R}_{v} \right] + (k_c^2 r^2 - n^2) R = 0$$

$$= r \left[r \times \frac{d^2}{dr^2} R + \frac{d}{dr} R \right] + (k_c^2 r^2 - n^2) R = 0$$

$$= r^2 \frac{d^2}{dr^2} R + r \frac{d}{dr} R + (k_c^2 r^2 - n^2) R = 0$$

↓

This is the standard form of Bessel differential equation.

Solving the Bessel differential equation using Math

Since the const term has function of 'R' we need to use Method of Frobenius which assumes the solution is infinite

Polynomial power series.

$\frac{d^2}{dr^2}$ term & $\frac{d}{dr}$ terms have coeff of 'r' one is linear & another

square terms. Sum of square & linear coefficient may have something which form a ∞ series which converge at ∞ .

— Why Frobenius Method? What observations can be made from this equation?

For observations, let's divide the eqn by r^2

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{d}{dr} R + \left(k_c^2 - \frac{n^2}{r^2} \right) R = 0$$

Here $\frac{d^2}{dr^2}$ term has coeff 1

$\frac{d}{dr}$ term has coeff $1/r \Rightarrow$ this term changes

as we move from center of WG cross section to inner wall

as $r \uparrow$ ~~the term~~ $1/r$ term gets weaker.

Changes as $\frac{1}{r^2}$ term changes as $\frac{1}{r^2}$ in const part same way as we move radially outward, 'R' term gets weaker

The 'R' term & $\frac{d}{dr} R$ term vary with 'r' in different ways ($1/r, 1/r^2$) to accommodate both changes / model both changes a single function like exponential, sine, cosine cannot be used. It needs a power series model which contains variations of all possible coefficients like $1/r, 1/r^2, \dots$ Basically something like a Taylor or Maclaurin series which can define any arbitrary curves.

Now there's a limitation for defining this ODE solution using Taylor or power Maclaurin type of series.

at $r=0$ the eqn $\rightarrow \infty$ (singularity). To overcome this, method of Frobenius was invented which takes care of this singularity at origin ($r=0$). Also, Practically the amplitude of $\bar{G}/\bar{H}$ at center of WG cannot be infinite. It's a finite value!

Method of Frobenius

In this method, the solution takes form of power series

$$R(r) = \sum_{k=0}^{\infty} C_k r^{k+s}$$

$s \rightarrow$ will help us to make sure there's no singularity at origin

$$\frac{dR}{dr} = \sum_{k=0}^{\infty} C_k (k+s) r^{k+s-1}$$

$$\frac{d^2 R}{dr^2} = \sum_{k=0}^{\infty} C_k (k+s)(k+s-1) r^{k+s-2}$$

Substitute this in the Bessel differential equation to verify the equation

$$\sum_{k=0}^{\infty} C_k (k+s)(k+s-1) r^{k+s-2} + \frac{1}{r} \sum_{k=0}^{\infty} C_k (k+s) r^{k+s-1} + \left(k_c^2 - \frac{n^2}{r^2} \right) \sum_{k=0}^{\infty} C_k r^{k+s} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+s) r^{k+s-2} [k+s-1+1] + \left(k_c^2 - \frac{n^2}{r^2} \right) \sum_{k=0}^{\infty} C_k r^{k+s} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+s)^2 r^{k+s-2} + k_c^2 \sum_{k=0}^{\infty} C_k r^{k+s} - \frac{n^2}{r^2} \sum_{k=0}^{\infty} C_k r^{k+s} = 0$$

~~Can move in to $\sum$ as it's depen~~

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+s)^2 r^{k+s-2} + k_c^2 \sum_{k=0}^{\infty} C_k r^{k+s} - n^2 \sum_{k=0}^{\infty} C_k r^{k+s-2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_k \left[(k+s)^2 - n^2 \right] r^{k+s-2} + k_c^2 \sum_{k=0}^{\infty} C_k r^{k+s} = 0$$

~~For $k=0$~~ We need to find ' C_k ' & ' s ' term

$$\sum_{k=0}^{\infty} C_k$$

Finding ' s '

$$C_0 (s^2 - n^2) r^{s-2} + C_1 ((1+s)^2 - n^2) r^{s-1} + \sum_{k=2}^{\infty} C_k ((k+s)^2 - n^2) r^{k+s-2} + k_c^2 \sum_{k=0}^{\infty} C_k r^{k+s} = 0$$

$$+ k_c^2 \sum_{k=0}^{\infty} C_k r^{k+s} = 0$$

Simplify this term to get in $k+s-2$ form

$$\text{let } k+s-2 = m$$

$$k+s-2 = m \Rightarrow k = m-2$$

$$\text{let } k+2 = m$$

$$\left. \begin{array}{l} \text{if } k \rightarrow 0 \quad m \rightarrow 2 \\ m \rightarrow 0 \quad k \rightarrow -2 \end{array} \right\} k = m-2$$

$C_k \rightarrow C_{m-2}$
 term changes to $k_c^2 \sum_{m=2}^{\infty} C_{m-2} r^{m-2+s}$ let me simplify in better way
 $= k_c^2 \sum_{m=2}^{\infty} C_{m-2} r^{m-2+s}$

Terms $k_c^2 \sum_{k=0}^{\infty} C_k r^{k+s}$ we need to get term $k+s-2$

If we are shifting power value by '2' then backwards then summation should shift by '2' forwards

$\Rightarrow r^{k+s} \rightarrow r^{k+s-2}$ then $k=0 \rightarrow k=2$

term changes to $k_c^2 \sum_{k=2}^{\infty} C_{k-2} r^{k+s-2}$

Finally

$$\begin{aligned}
 & C_0 (s^2 - n^2) r^{s-2} + C_1 ((1+s)^2 - n^2) r^{s-1} + \sum_{k=2}^{\infty} C_k [(k+s)^2 - n^2] r^{k+s-2} \\
 & + k_c^2 \sum_{k=2}^{\infty} C_{k-2} r^{k+s-2} = 0 \\
 \Rightarrow & \underbrace{C_0 (s^2 - n^2) r^{s-2}}_{\text{Indicial eqn}} + \underbrace{C_1 ((1+s)^2 - n^2) r^{s-1}}_{\text{Odd term}} + \underbrace{\sum_{k=2}^{\infty} [C_k ((k+s)^2 - n^2) + k_c C_{k-2}] r^{k+s-2}}_{\text{Recurrence relation}} = 0
 \end{aligned}$$

~~Co term cannot be zero~~ To make this eqn zero, every coefficient of the polynomial of the exponential r^x should be zero

$C_0 (s^2 - n^2) = 0$ $C_0 \neq 0 \rightarrow C_0$ refers to fundamental Amplitude & if it is 0 $\Rightarrow$ there no wave in WG

$\therefore s^2 - n^2 = 0 \Rightarrow s^2 = n^2$ $s = \pm n$ if $s = -n$ then $r^{s-2} \rightarrow$ diverges $\} \underline{s = +n}$ esp at $r=0$

$C_1 ((1+s)^2 - n^2) = 0$ ~~$C_1 \neq 0$~~ Same reason as above

~~$(1+s)^2 = n^2$~~ $C_1 \rightarrow$ next Spatial harmonic

It can/cannot exist

$\therefore$ either $C_1 = 0$ or $(1+s)^2 - n^2 = 0$

If $(1+s)^2 - n^2 = 0 \quad n^2 = (1+s)^2$

n^2 cannot be $(1+s)^2$ & s^2 at the same time

So this term $\neq 0$

$\therefore Q = 0$

Terms $\rightarrow C_k [(k+s)^2 - n^2] + C_{k-2} k^2 = 0$

Substitute $s = n$ ($\because s^2 = n^2 \quad s = \pm n$) why $-n$ neglected?
See the justification near Co term off derivation.

$$C_k [(k+n)^2 - n^2] + C_{k-2} k^2 = 0$$

$$C_k [k^2 + n^2 + 2nk - n^2] + C_{k-2} k^2 = 0$$

$$C_k k^2 + 2nk C_k + C_{k-2} k^2 = 0$$

$$C_k = \frac{-k^2}{k^2 + 2nk} C_{k-2} \quad \text{for } k \geq 2$$

Observe For $k=3$ we get $\frac{-k^2}{k^2 + 2n(3)} C_1$

$\therefore Q=0 \Rightarrow C_3=0$ Same for $k=5$ we get $C_{k-2}=C_3$

$\therefore C_5=0$

Finally all odd terms are $=0$

$$\therefore C_k = \frac{-k^2}{k(k+2n)} C_{k-2} \quad \text{for } k \geq 2 \text{ \& } k\text{-even}$$

$\rightarrow$ This is way the radial ^{component} E-field varies

Let's find C_0 using C_2

$$C_0 = \frac{-k_c^2}{2(2+n)} C_2$$

$$C_2 = \frac{-k_c^2}{4(1+n)} C_0$$

$$C_4 = \frac{-k_c^2}{4(4+2n)} C_2 = \frac{-k_c^2}{8(2+n)} C_2$$

$$C_4 = \frac{-k_c^2}{8(2+n)} \times \frac{-k_c^2}{4(1+n)} C_0 = \frac{k_c^4}{32(n+1)(n+2)} C_0$$

observe the pattern

$k=2m$
used for
even numbers

$$4 \times 8 = 32$$

$$\downarrow$$

$$4 \times 2^3 = 32$$

let's take $k=2m$
∴ k is even

$$C_{2m} = \frac{(-1)^n k_c^{2m}}{(n+1)(n+2) \dots (n+2m) \times \underline{\hspace{1cm}}}$$

$m=1 \rightarrow 4$
 $m=2 \rightarrow 32 \rightarrow 8 \text{ times } 2m$
 $m=3 \rightarrow 32 \times 2m \times m \times 4 \times 3 \times 32$
 $4 \times 3 \times 2 \times 16$
 $(n+1) \times 2$

$m=1 \rightarrow 4$
 $m=2 \rightarrow 4 \times 2 \times 2m = 32$
 $m=3 \rightarrow 32 \times 2 \times 6$
 2×3
 2×2^6
 64

$$C_6 = \frac{-k_c^2}{6(6+2n)} \times \frac{k_c^4}{32(n+1)(n+2)} C_0$$

$\downarrow$
 $2m$

$$\frac{-k_c^6}{6 \times 2 \times (n+3)} C_0 \times \frac{32(n+1)(n+2)}{\underline{\hspace{1cm}}}$$

$32 \times 6 \times 2$
 $3 \times 2 \times 1$
 $\frac{2m}{2} \times 64 \times 32 \times 1 \rightarrow m!$

So the missing pattern is $2^{2m} \times m!$

$$C_{2m} = \frac{(-1)^m k_c^{2m}}{2^{2m} \times m! (n+1)(n+2) \dots (n+2m)} \times C_0$$

Going back to ∞ Series

$$R(r) = \sum_{k=0}^{\infty} C_k r^{k+s} \quad k=2m \quad s=n$$

$$R(r) = \sum_{2m=0}^{\infty} \frac{(-1)^m k_c^{2m}}{2^{2m} \times m! (n+1)(n+2) \dots (n+2m)} \times r^{2m+n} \times C_0$$

I
Kind Bessel function

$$R(r) = \sum_{2m=0}^{\infty} C_{2m} r^{2m+s} \rightarrow \text{This is the Bessel function of first kind}$$

Let us write Bessel function of first kind actual form

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (n+m)!} \times \left(\frac{x}{2}\right)^{2m+n}$$

Let's refine $R(r)$ in $J_n(x)$ form

$\left(\frac{x}{2}\right)^{2m+n} \rightarrow \frac{k_c^{2m} r^{2m+n}}{2^{2m} \times m! (n+m)!} \times C_0$
 Sleepy!
 Tomorrow Freshmind
 $C_0 \rightarrow \frac{1}{2^{2m+n} \times k_c^{2m}} \times \left(\frac{1}{2}\right)^{2m+n}$

Target is to find 'Co' & write $R(r)$ as $J_n(x)$

$$R(r) = \sum_{m=0}^{\infty} \frac{(-1)^m k_0^{2m}}{2^{2m} m! (n+1)(n+2) \dots (n+2m)} \times r^{2m+n} \times C_0$$

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (n+m)!} \times \left(\frac{x}{2}\right)^{2m+n}$$

'Co' is coeff @ $m=0$ Can be derived by finding 1 term of $R(r)$

In $R(r)$ when $m=0$,

$$\begin{aligned} (-1)^m &\rightarrow 1; & k_0^{2m} &\rightarrow 1; \\ r^{2m+n} &\rightarrow r^n & 2^{2m} &\rightarrow 1 & m! &\rightarrow 1 \\ (n+1)(n+2) \dots (n+2m) &\rightarrow \text{doesn't exist} \\ && && &\text{as factorial starts at } m=1 \\ &&&&&\text{so this product turns to } 1 \end{aligned}$$

$$R(r) \text{ I term} = C_0 \times r^n$$

$$\text{I term in } J_n(x) = m=0 \rightarrow n+m! \rightarrow n!$$

↳ This starts with $n=1$

$$\begin{aligned} m! = 0; & \left(\frac{x}{2}\right)^{2m+n} \rightarrow \left(\frac{x}{2}\right)^n \\ (-1)^m &\rightarrow 1 \end{aligned}$$

$$\Rightarrow J_n(x) \text{ I term} = \left(\frac{x}{2}\right)^n \times \frac{1}{n!}$$

$$C_0 \times r^n = \left(\frac{x}{2}\right)^n \times \frac{1}{n!}$$

Relating C_0 & x $\rightarrow$ Both are unknown

For this then $R(r)$ to look more similar to $J_n(x)$ format

$$R(r) = \sum_{2m=0}^{\infty} \frac{(-1)^m}{m! \times (n+1)(n+2) \dots (n+2m)} r^n \times \left(\frac{k_c r}{2}\right)^{2m} \times C_0$$

↓
To write this as $n+2m$!

$$\left\{ \begin{array}{l} n+m! \\ = n! \times (n+1)(n+2) \dots (n+m) \\ \text{or } m! \times (n+1)(n+2) \dots (m+n) \end{array} \right.$$

$$R(r) = \sum_{2m=0}^{\infty} \frac{(-1)^m \times n! \times 2^n}{m! (n+m)! k_c^n} \times \left(\frac{k_c r}{2}\right)^{2m+n} \times C_0 \quad \left(\frac{\text{mult \& div } k_c^n}{2^n} \right)$$

Now compare with Bessel function

$$\underline{x = k_c r} \quad \therefore J_n(x) \rightarrow J_n(k_c r) \text{ for } R(r)$$

$$\frac{C_0 \times n! \times 2^n}{k_c^n} = 1$$

$$C_0 = \frac{k_c^n}{2^n n!}$$

Finally I can write $R(r) = \left[\frac{C_0 2^n n!}{k_c^n} \right] J_n(k_c r)$

Now after a long battle lets go back to our target

$$E_z(r, \phi) = R(r) \cdot \phi(\phi)$$

$$E_z(r, \phi) = \left[\frac{C_0 2^n n!}{k_c^n} \right] J_n(k_c r) * A \cos(n\phi) + B \sin(n\phi)$$

↳ multiply

Side Note:

There's an interesting thing about Bessel function

$R(r)$ ~~has~~ is a Bessel function $R(r)$ should be $J_n(x) + Y_n(x)$ form

$Y_n(x)$ is Bessel func of II kind $\rightarrow$ this term we didn't consider & it vanished

' $Y_n(0)$ ' makes 'oo' amplitude $\rightarrow$ occurs when there is singularity at origin. WG has finite amplitudes at $r=0$ & this term cannot exist as $r=0$ is also a part of WG field solution.

In coax,



Field exist b/w annular part of inner &

outer conductor $\Rightarrow r=0$ term doesn't exist in field solution

$\therefore$ Bessel functions of II kind exist in fields of coax

when its excited with very very high freqs where higher order modes other than TEM like TE & TM exist.

Back to waveguide ' E_z '

Multiply $R(r)$ & $\phi(\phi)$ terms let $\frac{C_0 2^n n!}{k_c^n} = C_r r$ some const

For easy

holding we assumed

this

$$E_z(r, \phi) = A C_r J_n(k_c r) \cos n\phi + (B_r J_n(k_c r) \sin n\phi$$

This is Transverse Component

What about longitudinal component $E_z(z)$? TM wave has $E_z(z)$

Going Back to Scalar helmholtz eqn

What we have derived is 2D cross section $\vec{E}$ field with has

r, ϕ component

$$E_z(r, \phi, z) = e_z(r, \phi) \times z(z)$$

$\downarrow$

need to find this & plug in

$$\nabla^2 E_z + K^2 E_z = 0 \text{ - Scalar helmholtz eqn}$$

We get when we combine 2 curl eqns (maxwell)

$$\nabla^2 E_z = \underbrace{\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial E_z}{\partial r} \right]}_{\text{Solution } R(r)} + \underbrace{\frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2}}_{\text{Solution } \phi(\phi)} + \underbrace{\frac{\partial^2 E_z}{\partial z^2}}_{\text{Solution } z(z)} + k^2 E_z = 0$$

Comparing $\phi(\phi)$ & $z(z)$ Both have const coeff for PDE

$\therefore z(z)$ will also have cos & sin term as solution

$$z(z) = C \cos(\beta z) + D \sin(\beta z) \rightarrow \text{comes from } \frac{\partial^2 E_z}{\partial z^2} = -k^2 E_z$$

$\hookrightarrow$ comes from

$$\frac{d^2 z}{dz^2} = -k^2 E_z$$

↓
longitudinal
+ Transverse wave
number

Reasoning $\left(\begin{array}{l} \text{Since } z \text{ is longitudinal} \\ \text{So } k^2 \rightarrow \underline{\beta^2} \end{array} \right.$

$$\therefore \text{Now } E_z(r, \phi, z) = C_n J_n(k_c r) [A \cos(n\phi) + B \sin(n\phi)] \times [C \cos(\beta z) + D \sin(\beta z)]$$

~~$$C_n J_n(k_c r) [A C \cos(n\phi) \cos(\beta z) + A D \cos(n\phi) \sin(\beta z) + B C \cos(\beta z) \sin(n\phi) + B D \sin(n\phi) \sin(\beta z)]$$~~

Analyzing The wave flow

Now let's apply Wall Boundary condition

at $r=a$ if any 'wg' radius is 'a' my $E_{tm} = 0$

$\Rightarrow E_r \text{ component} = 0 \forall \phi$ (Transverse term)

$$E(r, \phi) = 0 \text{ at } r=a \quad C_n J_c(k_c a) \times A \cos(n\phi) + B \sin(n\phi) = 0$$

$\hookrightarrow$ term = 0

The Biting out of J_n

$$J_n(k_c a) = 0$$

$$J_n(k_c a) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+n)!} \left(\frac{k_c a}{2}\right)^{2m+n}$$

① $\rightarrow$ When does this term = 0 $\rightarrow$ alternate + & - Should cancel
 $\rightarrow \left(\frac{k_c a}{2}\right)^{2m+n}$ $\uparrow$ power term as $2m+n \uparrow$

[We are replacing 'm' with letter 'k' to use 'm' for a different meaning]

Don't confuse with 2m+n derivation part

$\rightarrow m! (m+n)! \rightarrow$ term $\downarrow$ the $J_n(x)$

Key point the rate of $\left(\frac{k_c a}{2}\right)^{2m+n}$ $\uparrow$ is less than how $m! (m+n)! \rightarrow$ increase makes $J_n(k_c a)$ decay with alternate + & - cancel out.

② $\rightarrow$ Will this becomes zero ($J_n(k_c a)$) once or many times?

$(-1)^m$ adds & subtract series of chunk as $k_c a$ grows radially out ($a \rightarrow \text{rad } r \rightarrow 0 \text{ to } a$)

So, sometimes they cancel off for some values of m & n combination.

They are called zero crossings.

Also, they ~~cancel~~ have multiple zero crossings indicate the wave pattern is like sinusoidal

Also since the Bessel function as explained above has

terms $m! (m+n)! \left(\frac{k_c a}{2}\right)^{2m+n}$ their growth overall is decaying (converge)

$\therefore$ Bessel function is a decaying sine wave as it moves radially outward.

Where will the zero crossings happen for Bessel?

For $\sin(\theta)$ we know it's $\pm n\pi$ $n=1, 2, \dots$

Similarly for Bessel it's not simple as decay element is added

So we name a variable for zero crossings

Let's call it P_{nm}

$n \rightarrow$ which order of Bessel function we are solving

$m \rightarrow$ which specific zero crossing we are looking at (1st root, 2nd root, ...) $\rightarrow m^{\text{th}} \text{ time } J_n(ka) = 0$

$n \rightarrow$ is determined by azimuthal wave $\cos(n\phi)$

$$\left(\text{number } C_n \underbrace{J_n(ka)}_r \left[A \cos(n\phi) + B \sin(n\phi) \right] r \dots \right)$$

$\downarrow \quad \quad \downarrow$
 $\phi \quad \quad \phi$

this 'n' term can also make product zero ($E_z = 0$)

So, To make $J_n(ka) = 0$ its root will be

~~$ka = \text{root}$~~

$$ka = P_{mn}$$

$$k_c = \frac{P_{mn}}{a}$$

— Cut off wave number

We know that ~~$k_c = \frac{2\pi f}{c}$~~ (wave number $\propto$ freq)

$$f = \frac{c}{\lambda}$$

~~$k_c = \frac{2\pi f}{c} \Rightarrow \text{Cut off frequency}$~~

$$f_c = \frac{P_{mn}}{2\pi a}$$

Remember? $k_c^2 = \omega^2 \mu \epsilon - \beta^2$

Cut off condition happens if wave loses ability to travel in W.G $\Rightarrow$ Longitudinal component $= 0$ Propagation $= 0$

$\Rightarrow$ longitudinal propagation const $\beta = 0$

$\Rightarrow k_c^2 = k^2 = \omega^2 \mu \epsilon$

Now $\omega \sqrt{\mu \epsilon} = \frac{P_{mn}}{a}$

$\omega = 2\pi f$

$f = f_c \Rightarrow$ cut off freq

$2\pi f_c \sqrt{\mu \epsilon} = \frac{P_{mn}}{a}$

$f_{c_{TM}} = \frac{P_{mn}}{2\pi a \sqrt{\mu \epsilon}}$

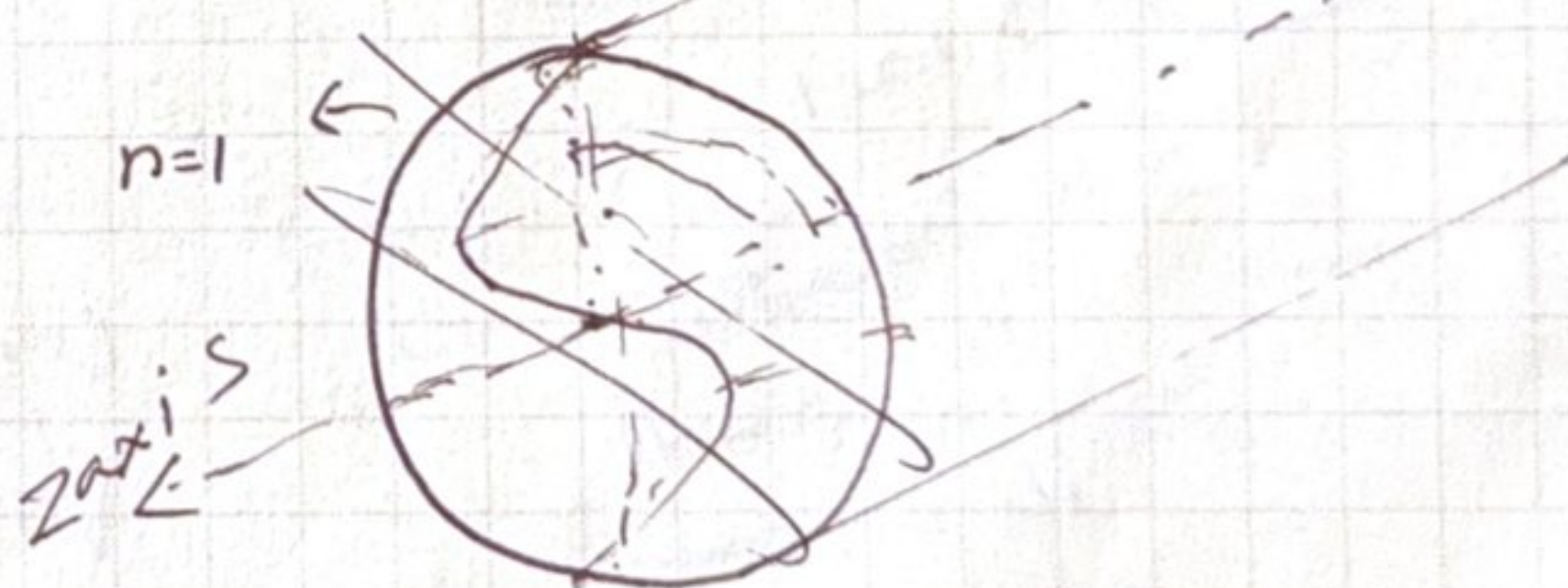
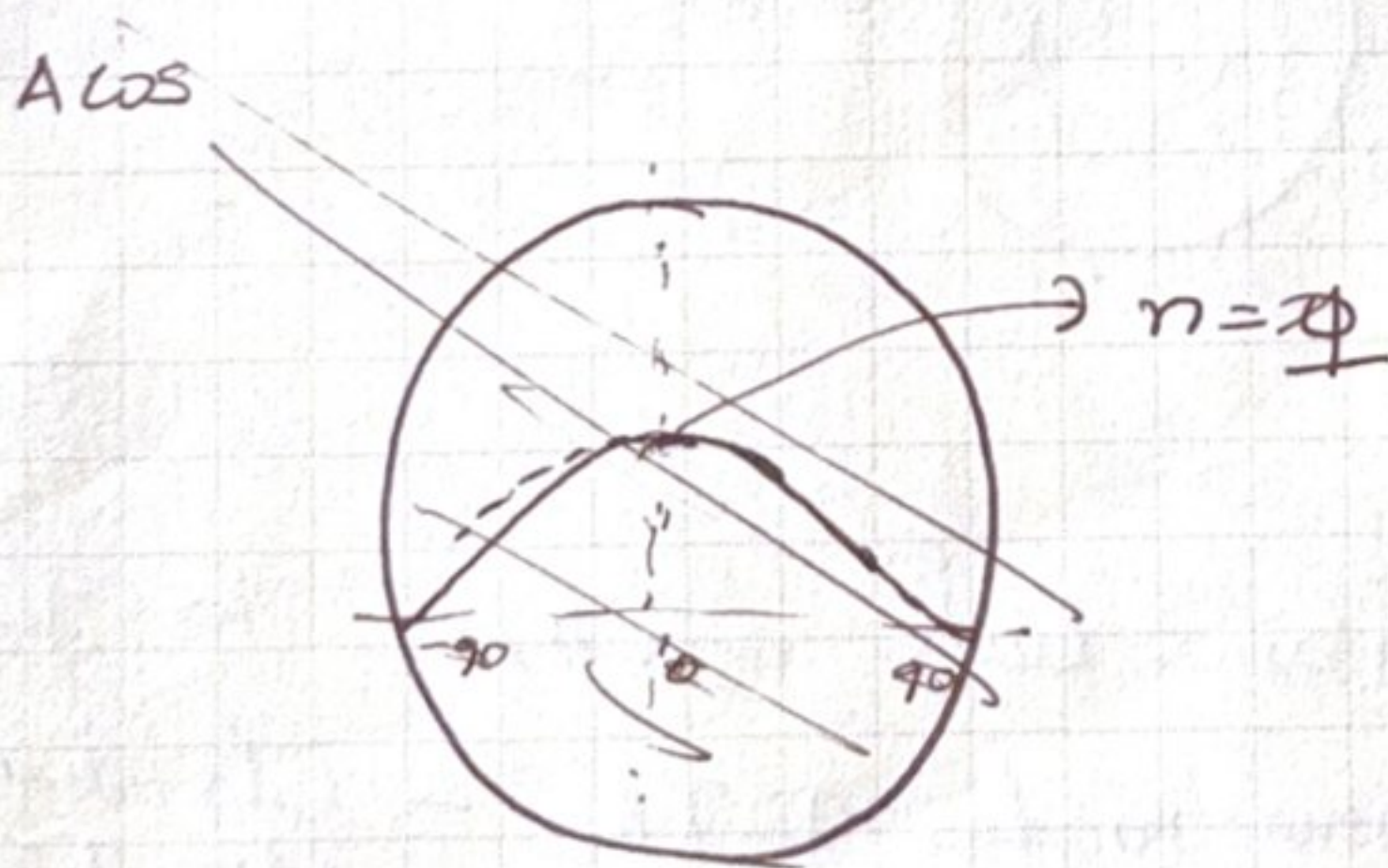
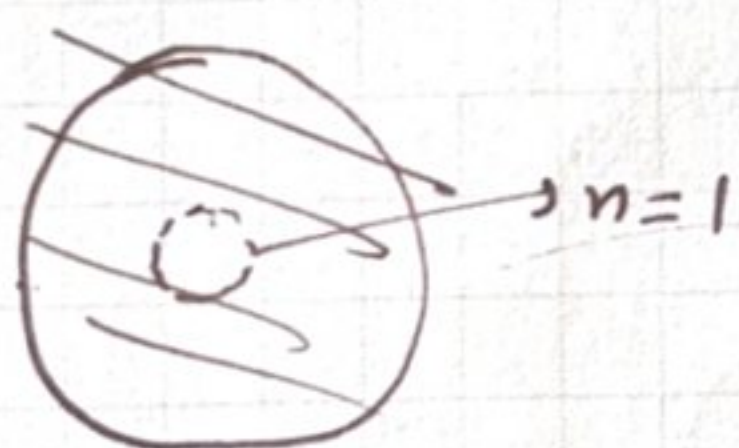
Where $\frac{1}{\sqrt{\mu \epsilon}}$ is the

Speed of wave in Vacuum ~~wave guide~~

Before concluding TM_{mn} propagation,

$n \rightarrow$ depends on ϕ (angular component of 'Ez')

Dictates num of full waves around the circumference of W.G (the conduit pipe)



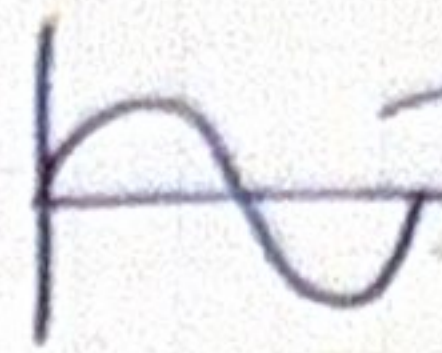
$n=1$



$\rightarrow z$

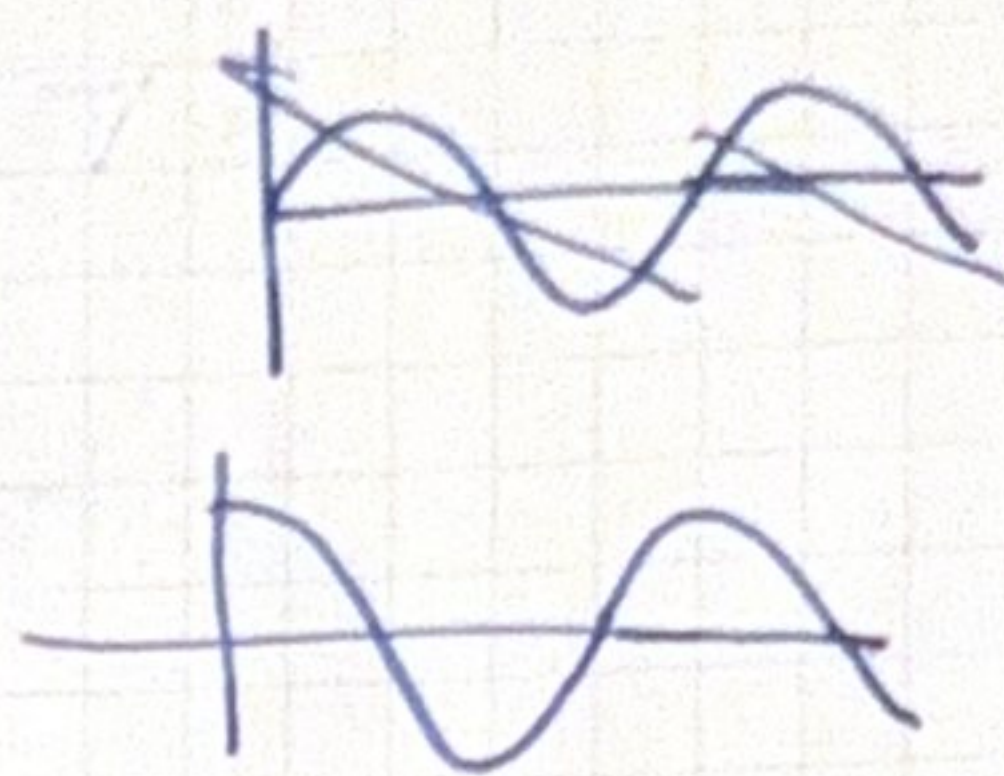
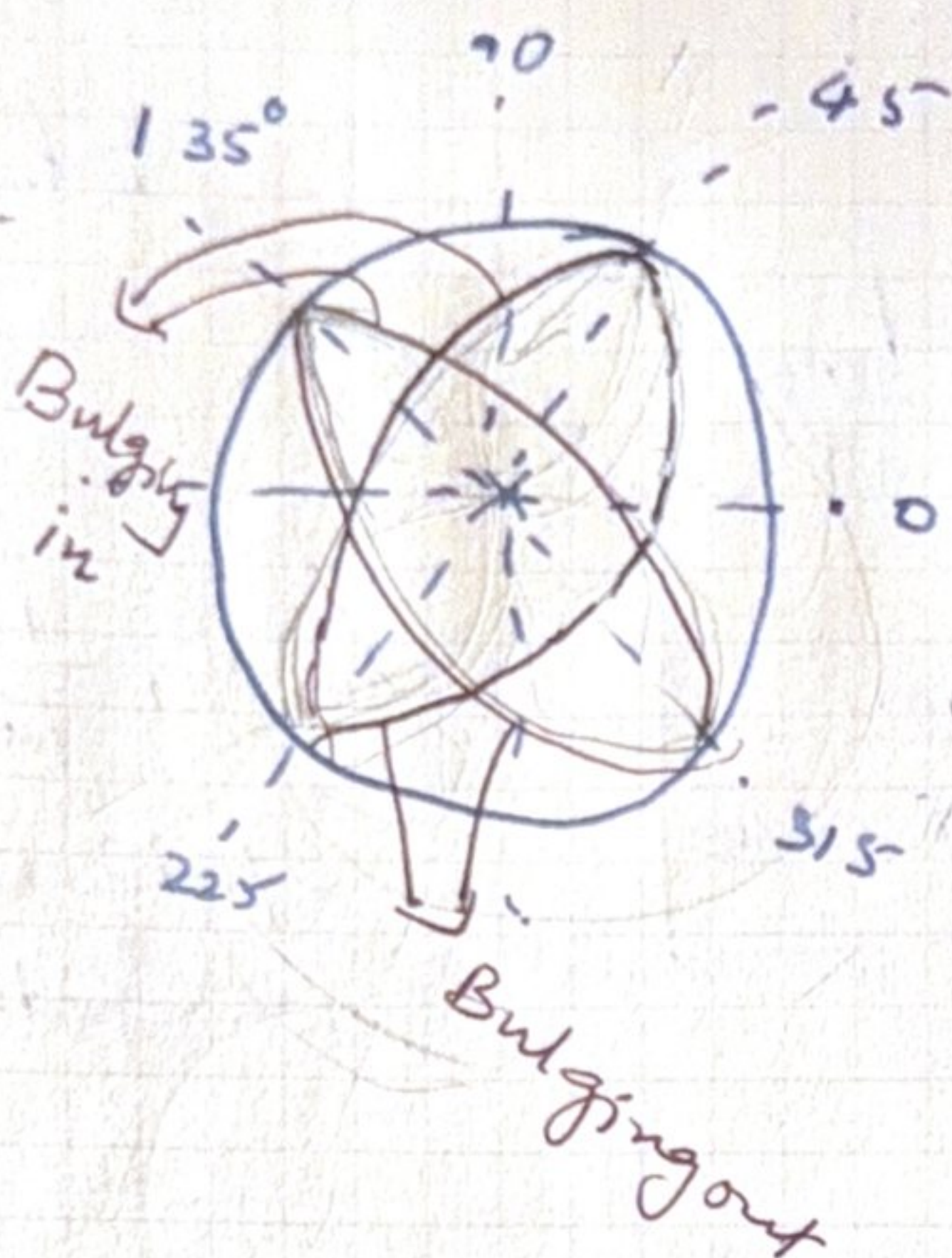


Bulging out along z axis



wrap this around circle to get $n=1$ shape (use $\cos$ wave)

$n=2$

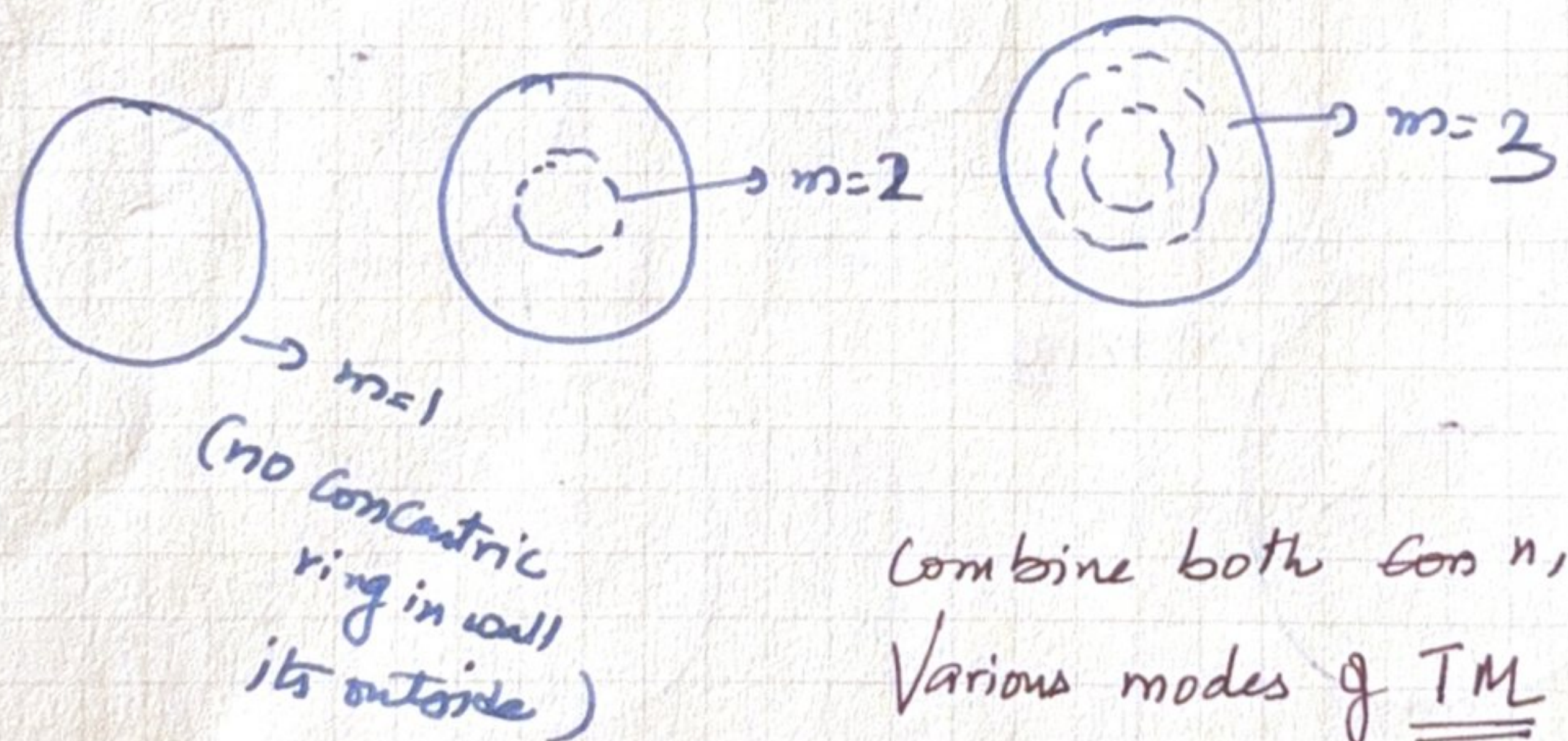


$n=2$



Coming to 'm'

'm' depends on specific zero crossing of Bessel function
(How many concentric rings exist b/w center & outer wall of WG)



Combine both for n, m to get various modes of TM - let's discuss after TE

Each TM_{nm} has different pattern & each mode (mostly few) are used for specific applications depending on where we want $\vec{E}$ & $\vec{H}$ field maxima & minima

TE Wave

This time $E_z = 0$, $H_z \neq 0$ Going back to $\vec{E}$ eqns derived ^{curl}

$$\cancel{\frac{1}{r}} \frac{\partial H_\phi}{\partial r} = \cancel{\frac{1}{r}} \frac{\partial H_r}{\partial \phi}$$

$$\frac{1}{r} \frac{\partial E_\phi}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} = -\mu j\omega H_z$$

$$E_r = -\frac{j}{k_c^2} \left[\frac{\omega\mu}{r} \frac{\partial H_z}{\partial \phi} \right]$$

$$E_\phi = -\frac{j}{k_c^2} \left[-\omega\mu \frac{\partial H_z}{\partial r} \right]$$

$$H_\phi = -\frac{j}{k_c^2} \left[\frac{\beta}{r} \frac{\partial H_z}{\partial \phi} \right]$$

$$H_r = -\frac{j}{k_c^2} \left[\beta \frac{\partial H_z}{\partial r} \right]$$

Same story repeat $\rightarrow$ forming Scalar Helmholtz eqn

This guy's responsible for messy func formation

$$r \frac{\partial E_\phi}{\partial r} = -\frac{j}{k_c^2} \left[-\omega\mu \frac{\partial^2 H_z}{\partial r^2} \right] + \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial r} \rightarrow$$

$$\frac{\partial E_r}{\partial \phi} = -\frac{j}{k_c^2} \left[\frac{\omega\mu}{r} \frac{\partial^2 H_z}{\partial \phi^2} \right] \quad \text{--- substitute in } H_z \text{ eqn}$$

$$\therefore \cancel{\frac{1}{r}} \frac{j}{k_c^2} \left[\omega\mu \frac{\partial^2 H_z}{\partial r^2} \right] + \frac{1}{r} \frac{j}{k_c^2} \left[\frac{\omega\mu}{r} \frac{\partial^2 H_z}{\partial \phi^2} \right] = \cancel{j\omega\mu H_z}$$

$$\therefore \frac{j}{k_c^2} \left[\omega\mu \frac{\partial^2 H_z}{\partial r^2} \right] + \frac{j\omega\mu}{r k_c^2} \frac{\partial H_z}{\partial r} + \frac{1}{r} \frac{j}{k_c^2} \left[\frac{\omega\mu}{r} \frac{\partial^2 H_z}{\partial \phi^2} \right] = \cancel{j\omega\mu H_z}$$

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \phi^2} + k_c^2 H_z = 0 \quad \text{--- This is looking like Scalar Helmholtz eqn}$$

The above eqn is helmholtz eqn with only transverse component of wave propagation.

Let us include the longitudinal component too & solve full 3D PDE. — (Raising difficulty for fun)

We know that $k^2 = k_c^2 + \beta^2$

$\downarrow$ $\downarrow$ $\rightarrow$ longitudinal wave number

$\downarrow$ transverse wave number

Overall propagation wave number

In our helmholtz eqn where did ' β^2 ' term disappear?

$$k_c^2 = \omega^2 \mu \epsilon - \beta^2$$

$$\Rightarrow k = \omega \sqrt{\mu \epsilon}$$

$$\frac{\partial^2 H_2}{\partial r^2} + \frac{1}{r} \frac{\partial H_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_2}{\partial \phi^2} + K_c^2 H_2 = 0$$

How to turn 'Kc' to 'K' Carefully?

$$\frac{\partial^2 H_2}{\partial r^2} + \frac{1}{r} \frac{\partial H_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_2}{\partial \phi^2} + k^2 H_2 - \beta^2 H_2 = 0$$

We previously assumed $\frac{\partial}{\partial z}$ as $-j\beta$

$\frac{\partial}{\partial z} \rightarrow -j\beta$
 $\frac{\partial^2}{\partial z^2} \rightarrow (-j\beta)^2 = -\beta^2$

Remember Fourier transform

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2} + k^2 H_z = 0$$

this is full PDE ~~of helmholz~~ laplacian of H_2 & $K^2 H_2$ combination
full 3D helmholz eqn

once again, using Frobenius method for solving $\frac{\partial}{\partial r}$ section

before that Assuming this PDE as $R \phi z$ & solving using Variable Separable method

$$\text{Let } H_z(r, \phi, z) = R(r) \phi(\phi) z(z)$$

$$\frac{\partial}{\partial r} H_z = \left[\frac{\partial}{\partial r} (R(r)) \right] \times \phi(\phi) z(z)$$

$$\frac{\partial^2}{\partial r^2} H_z = \left[\frac{\partial^2}{\partial r^2} R(r) \right] \times \phi(\phi) z(z)$$

$$\frac{\partial^2 H_z}{\partial \phi^2} = R(r) \frac{\partial^2}{\partial \phi^2} [\phi(\phi)] z(z)$$

$$\frac{\partial^2 H_z}{\partial z^2} = R(r) \frac{\partial^2}{\partial z^2} (z(z)) \phi(\phi)$$

Substitute these in Helmholtz eqn

$$\begin{aligned} & \left[\frac{\partial^2}{\partial r^2} (R(r)) \times \phi(\phi) z(z) \right] + \frac{1}{r} \left[\frac{\partial}{\partial r} (R(r)) \times \phi(\phi) z(z) \right] \\ & + R(r) z(z) \frac{\partial^2}{\partial \phi^2} [\phi(\phi)] + k^2 R(r) \phi(\phi) \frac{\partial^2}{\partial z^2} (z(z)) \\ & + k^2 (R(r) \phi(\phi) z(z)) = 0 \end{aligned}$$

$$\text{Let } R(r) \rightarrow R \quad \phi(\phi) \rightarrow \phi \quad z(z) \rightarrow z$$

$$\phi z \frac{\partial^2}{\partial r^2} (R) + \frac{\phi z}{r} \frac{\partial R}{\partial r} + \frac{R z}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} + R \phi \frac{\partial^2 z}{\partial z^2} + k^2 R \phi z = 0$$

Divide by $R \phi z$ both sides

$$\frac{1}{R} \frac{\partial^2}{\partial r^2} (R) + \frac{1}{R r} \frac{\partial R}{\partial r} + \frac{1}{r^2 \phi} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{1}{z} \frac{\partial^2 z}{\partial z^2} = -k^2$$

longitudinal part

this PDE will be equal to const only when all these ^{func} R, ϕ, z are functions equal to a constant (each individual sum term is const)

longitudinal
transverse
part

Finally
$$H_z = \left[\frac{C_0 2^n n!}{k_c^n} \right] J_n(k_c r) \times \left[A \cos(n\phi) + B \sin(n\phi) \right] \times \left[C \cos(pz) + D \sin(pz) \right]$$

Again Applying Boundary conditions for predicting wave propagation along waveguide.

My waveguide radius is again 'a'

For conductor to die boundary for H field

Conductor boundary H_{tan} creates Surface Current

$$\vec{J}_s = \hat{n} \times H_{tan} \Rightarrow 'H_z' \text{ is non zero at conductive boundary}$$

which term will constraint wave & make it zero at boundaries

for propagate in oscillatory mode?

We have to rely on ' E_{tan} ' again coz $E_{tan} = 0$ at boundary.

Relation between E_{tan} & H_{tan} in TE wave

$$\frac{1}{r} \frac{\partial E_\phi}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} = -\mu_j \omega H_z$$



this term = 0 @ $r=a$

normal field at wall boundary if no ω
tan field at wall boundary of ω

$$H_z = H_{tan} + H_{norm}$$

Equating tan fields

$$\frac{1}{r} \frac{\partial E_\phi}{\partial r} = -\mu_j \omega H_{tan}$$

out of E_r E_ϕ E_z , For TE wave, $E_z = 0$ E_r is normal to surface E_ϕ is tangential to surface of w.g.

E_ϕ term tends to zero @ boundary & E_r term remains

We already formed equation

$$E_\phi = \frac{-j}{k_c^2} \left[-\omega\mu \frac{\partial H_z}{\partial r} \right]$$

$$\text{at } r=a \quad E_\phi = 0$$

$$0 = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial r}$$

This only tends to 0 when $\frac{\partial H_z}{\partial r} \Big|_{r=a} = 0$

We already derived H_z & if we differentiate with 'r', only

$J_c(k_c r)$ is func of r

This func when $r=a$ should tend to 0

$$\frac{\partial H_z}{\partial r} \Big|_{r=a} \propto \frac{\partial J_c(k_c r)}{\partial r} \Big|_{r=a} = 0$$



This will be zero @ roots of Bessel function

Let that root be P'_{mn} a const

$$k_c a = P'_{mn}$$

$$k_c = \frac{P'_{mn}}{a}$$

$$k_c = \frac{2\pi f_c}{c}$$

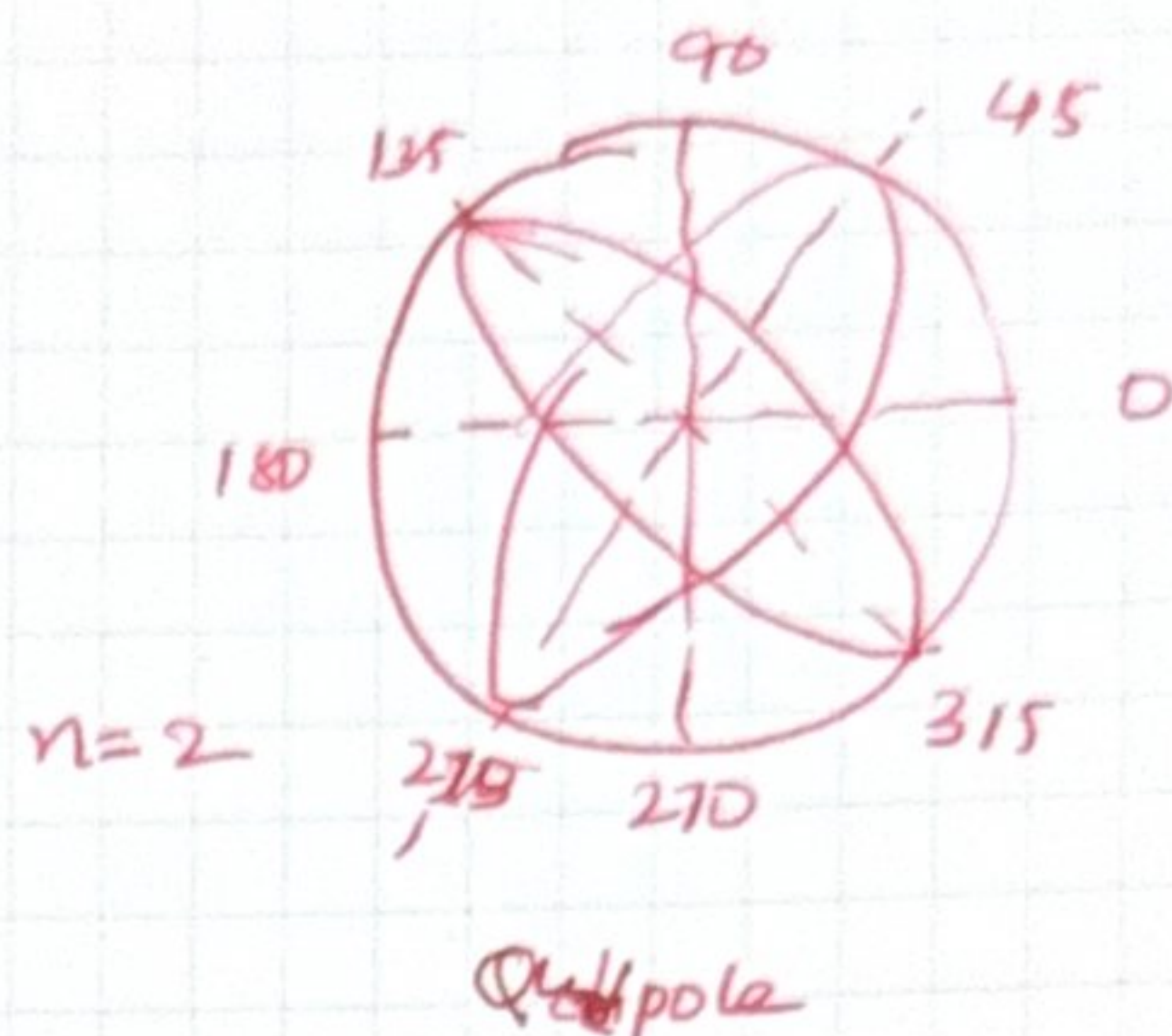
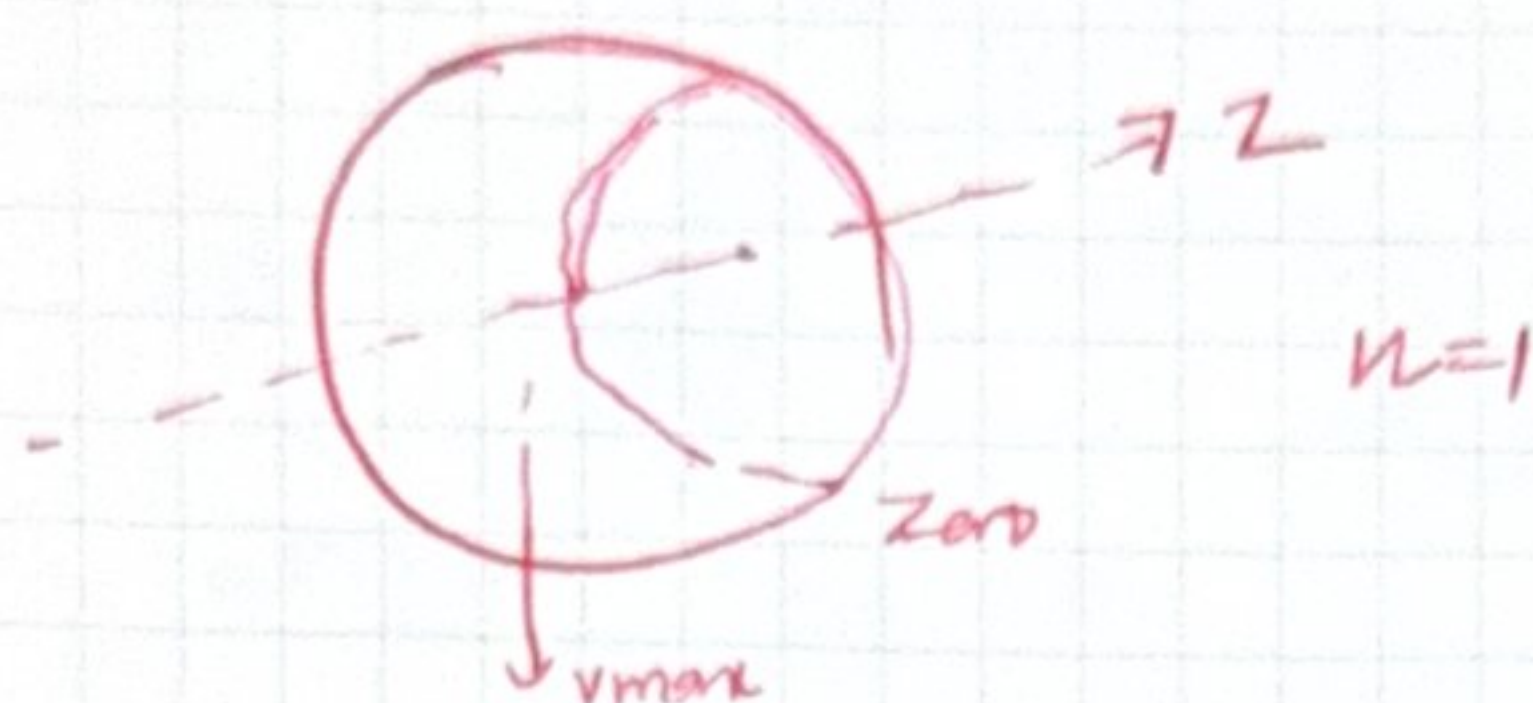
$$\therefore \boxed{f_{c, TE_{m,n}} = \frac{c \times P'_{mn}}{2\pi a}}$$

TE_{m,n} Propagation

$n \rightarrow$ depends on $\phi \rightarrow$ dictates number of full waves around
 $\hookrightarrow$ Angular component of H_z
 the waveguide

Same like TM wave but for H_z

H_z

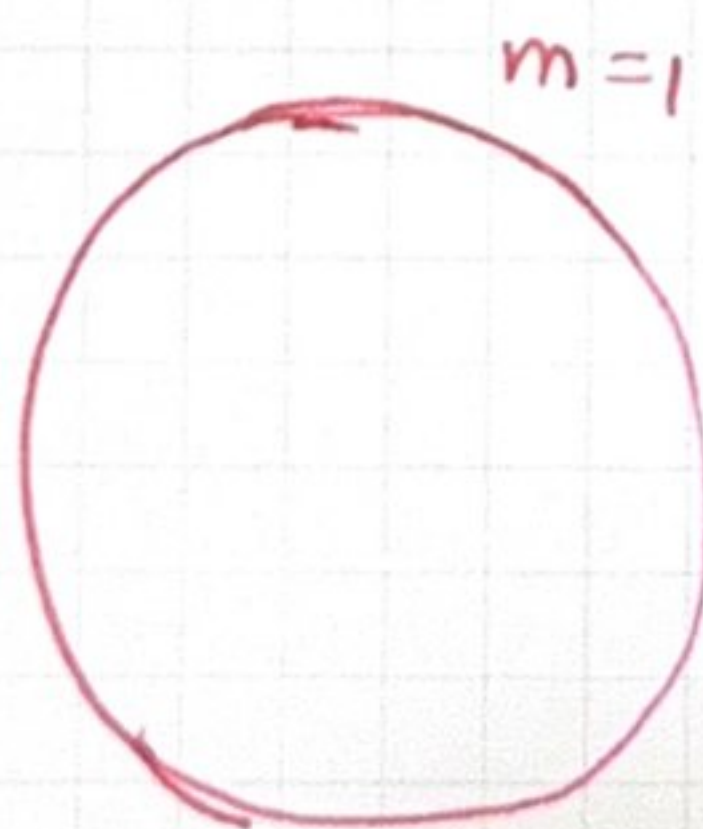
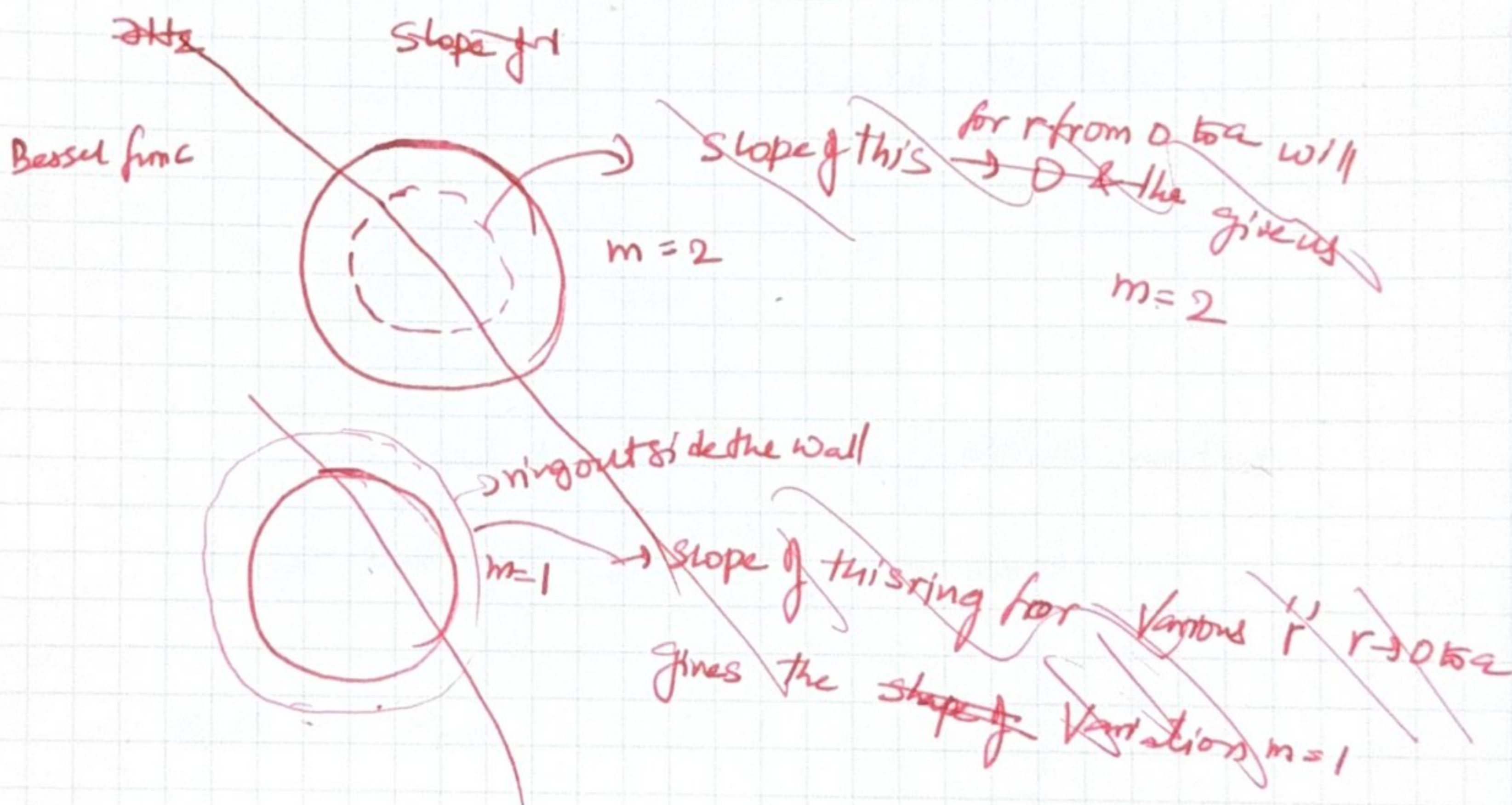


$n=3$ ~~drop pole~~ ^{Six} $\Rightarrow$ full cycles

$m \rightarrow$ This is where TE & TM differs

For TE waves Slope of $H_z \rightarrow 0$ at wall

For TM $\rightarrow E_z \rightarrow 0$ at wall



This is slope of H_z